Example 1.1

$f(x) = x\tan^{-1}x = x\arctan x$

Taylor series for $\arctan x$ at point 0.

$f(x) = f(0) + \frac{f'(0)}{1}x + \frac{f''(0)}{2!}x^2 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$

$\arctan 0 = 0.$

$(\arctan x)'\Big|_{x=0} = \frac{1}{1+x^2}\Big|_{x=0} = 1.$

$(\arctan x)''\Big|_{x=0} = \frac{-2x}{(1+x^2)^2}\Big|_{x=0} = 0.$

$(\arctan x)'''\Big|_{x=0} = -(3-1)! = -2!$

$(\arctan x)^{(4)}\Big|_{x=0} = 0.$

$(\arctan x)^{(5)}\Big|_{x=0} = 4!$

then $\arctan x = 0 + x + 0 - \frac{2!}{3!}x^3 + 0 + \frac{4!}{5!}x^5 \cdots$

$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{n=0}^{\infty}(-1)^n \cdot \frac{x^{2n+1}}{2n+1}$

Hence $x\arctan x = \sum_{n=0}^{\infty}(-1)^n \cdot \frac{x^{2n+2}}{2n+1}.$

select (e)

Example 1.2.

$f(x) = (2+x^2)^{-2}$

When $x=0$, $f(x) = \frac{1}{4}$.

$x=0$. (a) $= \frac{1}{4}$ (b) $= \frac{1}{2}$ (c) $= 0$

(d) $= -\frac{1}{2}$ (e) 0.

We select (a)!

Example 1.3.

By Taylor Series at a

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$(\ln x)' = \frac{1}{x}$

$(\ln x)'' = -\frac{1}{x^2}$

$(\ln x)''' = \frac{1\cdot 2}{x^3}$

$(\ln x)^{(4)} = -\frac{2\cdot 3}{x^4}$

$\vdots$

$(\ln x)^{(n)} = (-1)^{n+1} \cdot \frac{(n-1)!}{x^n}$

When $x=3$.

$f^{(n)}(3) = (-1)^{n+1} \cdot \frac{(n-1)!}{3^n}$

Then $f(x) = \ln 3 + \sum_{n=1}^{\infty} \frac{f^{(n)}(3)}{n!}(x-3)^n$

$= \ln 3 + \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{(x-3)^n}{n \cdot 3^n}$

(b)!

Example 2-1.

$$(x\ln x)\cdot\frac{dy}{dx}+y=x^2e^x$$

$$\frac{dy}{dx}+\frac{y}{x\ln x}=\frac{xe^x}{\ln x}$$

$$P(x)=\frac{1}{x\ln x},\quad Q(x)=\frac{xe^x}{\ln x}.$$

$$I(x)=e^{\int P(x)\,dx}=e^{\int\frac{1}{x\ln x}dx}=e^{\int\frac{1}{\ln x}\cdot\frac{1}{x}dx}$$ let $u=\ln x.$

$$=e^{\int\frac{1}{u}du}=e^{\ln u}=e^{\ln(\ln x)}=\ln x.$$

$$y(x)=\frac{1}{I(x)}\left[\int I(x)\,Q(x)\,dx+C\right]$$

$$=\frac{1}{\ln x}\cdot\left[\int\ln x\cdot\frac{xe^x}{\ln x}\,dx+C\right]$$

$$=\frac{1}{\ln x}\left[\int xe^x\,dx+C\right]$$

$$=\frac{xe^x-e^x+C}{\ln x}$$

$$y(e)=\frac{e^{e+1}-e^e+C}{1}=1\quad\Rightarrow\quad C=1-e^{e+1}+e^e.$$

Then $$y(x)=\frac{e^x(x-1)+1-e^e(e-1)}{\ln x}.$$ select (d)

Example 1.4.

Binomial Series

$$(1+x)^{\frac{1}{3}} = \sum_{n=0}^{\infty} \binom{\frac{1}{3}}{n} x^n$$

Taylor Series at 0.

$$(1+x)^{\frac{1}{3}}\Big|_{x=0} = 1$$

$$((1+x)^{\frac{1}{3}})'\Big|_{x=0} = \frac{1}{3}(1+x)^{-\frac{2}{3}}\Big|_{x=0} = \frac{1}{3}$$

$$((1+x)^{\frac{1}{3}})''\Big|_{x=0} = -\frac{2}{3^2}(1+x)^{-\frac{5}{3}} = -\frac{2}{3^2}$$

$$((1+x)^{\frac{1}{3}})'''\Big|_{x=0} = \frac{2\times5}{3^3} \qquad ((1+x)^{\frac{1}{3}})^{(4)}\Big|_{x=0} = -\frac{2\times5\times8}{3^4}$$

We can find the general term

$$(-1)^{n-1} \cdot \frac{2\times5\times\cdots(3n-4)}{3^n}$$

Then

$$(1+x)^{\frac{1}{3}} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} (-1)^{n-1} \frac{2\cdot5\ldots(3n-4)}{3^n\cdot n!} x^n$$

$$\binom{\frac{1}{3}}{n} = (-1)^{n-1} \cdot \frac{2\cdot5\cdot8\cdots(3n-4)}{3^n\cdot n!}$$

Select (b)

Example 2.2.

Differentiate $y = \frac{1}{(x+k)^3}$,

$dy = d((x+k)^{-3}) = -3(x+k)^{-4}\,dx$

$\frac{dy}{dx} = -3(x+k)^{-4}$

$y^{\frac{1}{3}} = (x+k)^{-1} \Rightarrow y^{\frac{4}{3}} = (x+k)^{-4}$

$\Rightarrow \frac{dy}{dx} = -3y^{\frac{4}{3}}$

To be vertical to the curve $\frac{dy}{dx} = \frac{1}{3y^{\frac{4}{3}}}$

$\Rightarrow 3y^{\frac{4}{3}}dy = dx$

Integrate both sides.

$3\int y^{\frac{4}{3}}dy = x + C$

$\frac{9}{7}y^{\frac{7}{3}} = x + C$

$y = (\frac{7}{9}x + C)^{\frac{3}{7}}$.

select (e)

Example 2-3

$$x^2 \frac{dy}{dx} + 2xy = \cos x$$

$$\frac{dy}{dx} + \frac{2y}{x} = \frac{\cos x}{x^2}$$

$$P(x) = \frac{2}{x}, \quad Q(x) = \frac{\cos x}{x^2}$$

$$I(x) = e^{\int p(x)\,dx} = e^{2\ln x} = x^2$$

$$y(x) = \frac{1}{I(x)} \left[\int I(x) Q(x)\,dx + C \right]$$

$$= \frac{1}{x^2} \left[\int x^2 \cdot \frac{\cos x}{x^2}\,dx + C \right]$$

$$= \frac{1}{x^2} (\sin x + C)$$

Since $y(\pi) = 0$.

$$y(\pi) = \frac{1}{\pi^2} (\sin \pi + C) = 0$$

$$C = 0.$$

then $y(x) = \dfrac{\sin x}{x^2}$

select (e)

Example 3.1.

$x = \sqrt{t}$, $y = \sin t$. $0 \le t \le 2\pi$.

$y = \sin x^2$

$\frac{dy}{dx} = 2x \cdot \cos x^2 = 0$

$x = 0$.

$x^2 = \frac{\pi}{2}, \frac{3}{2}\pi$.

$x = 1.25$ and 2.17 local maximum or minimum.

select (b)

Example 3.2.

$x = 1 + e^{2t}$, $y = e^t$,

$x = 1 + y^2$.

select (d)

Example 3.3.

$x = 2\cos t$, $y = 1 + \sin t$. Since $\sin^2 t + \cos^2 t = 1$.

$(\frac{x}{2})^2 + (y-1)^2 = 1$.

select (a)

Example 4.1

$y' = y(x-2)(x-3)$.

From the graphs, $y > 0$.

then. when $x < 2$, $y' > 0$.

$2 < x < 3$, $y' < 0$

$x > 3$ $y' > 0$.

select (b).

Example 4.2.

When $t = 0$ $x = f(0) = -1$, $y = g(0) = 0$. $(-1, 0)$.

$t = 1$ $x = f(1) = 0$. $y = g(1) = 0$ $(0, 0)$.

When $t = -2$. $x = f(-2) = a > 2$, $y = g(-2) = b < -2$. (a, b).

select (b).

Example 5.1

About y-axis

$$S = \int_1^2 2\pi x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$\frac{dy}{dx} = \frac{3x^2}{6} - \frac{x^{-2}}{2} = \frac{x^2}{2} - \frac{1}{2x^2}$$

$$\sqrt{1+\left(\frac{x^2}{2}-\frac{1}{2x^2}\right)^2} = \sqrt{\left(\frac{x^2}{2}+\frac{1}{2x^2}\right)^2} = \frac{x^2}{2}+\frac{1}{2x^2}$$

$$S = 2\pi \int_1^2 x \left(\frac{x^2}{2}+\frac{1}{2x^2}\right) dx$$

$$= 2\pi \int_1^2 \left(\frac{x^3}{2}+\frac{1}{2x}\right) dx$$

$$= \pi \int_1^2 \left(x^3+\frac{1}{x}\right) dx$$

$$= \pi \left(\frac{x^4}{4}+\ln x\right)\Big|_1^2$$

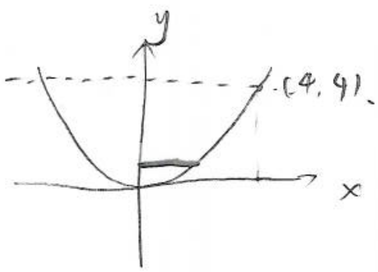
$$= \pi\left(\frac{2^4-1}{4}+\ln 2\right)$$

$$= \pi\left(\frac{15}{4}+\ln 2\right)$$

select (a)

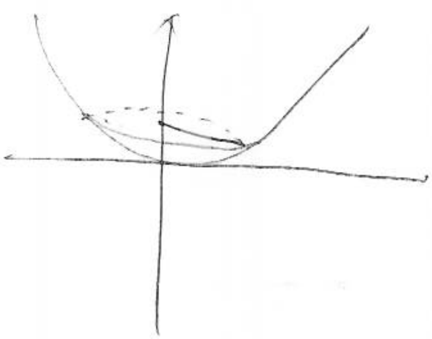
Example 5.2.

Assume $y = ax^2$.



$$4 = a \cdot 4^2 \Rightarrow a = \frac{1}{4}.$$

$$y = \frac{x^2}{4}.$$



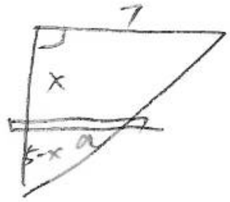
$$r = \sqrt{4y}.$$

Size of the circle area is $\pi r^2 = 4\pi y$.

$0 \leq y \leq 4$.

$$V = \int_0^4 4\pi y \, dy = 2\pi y^2 \Big|_0^4 = 32\pi.$$

Example 5.3



$$\frac{a}{5-x} = \frac{7}{5}$$

$$\Rightarrow \quad a = 7 - \frac{7}{5}x.$$

$$P_i = \rho g d = \rho g (x_i + 2)$$

Force on small fraction of the plate

$$F_i = P_i A = P_i (a \Delta x_i) = \rho g (x_i + 2)(7 - \tfrac{7}{5}x_i)\Delta x_i$$

$$F = \Sigma F_i.$$

In integral.

$$F = \int_0^5 \rho g (x+2)(7 - \tfrac{7}{5}x)\, dx = \frac{385}{6}\rho g$$

select (b)